

# Frobenius manifolds on orbits spaces

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## Abstract

The orbits space of an irreducible linear representation of a finite group is a variety whose coordinate ring is the ring of invariant polynomials. Boris Dubrovin proved that the orbits space of the standard reflection representation of an irreducible finite Coxeter group  $\mathcal{W}$  acquires a natural polynomial Frobenius manifold structure. We apply Dubrovin's method on various orbits spaces of linear representations of finite groups. We find some of them has non or several natural Frobenius manifold structures. On the other hand, these Frobenius manifold structures include rational and trivial structures which are not known to be related to the invariant theory of finite groups.

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# 1 Introduction

Frobenius manifold is a geometric realization introduced by B. Dubrovin for a potential satisfying a system of partial differential equations known as Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations which describes the module space of two dimensional topological field theory. Remarkably, Frobenius manifolds are also recognized in many other fields in mathematics like invariant theory, quantum cohomology, integrable systems and singularity theory [5]. Briefly, a Frobenius algebra is a commutative associative algebra with identity  $e$  and a nondegenerate bilinear form  $\Pi$  compatible with the product, i.e.,  $\Pi(a \circ b, c) = \Pi(a, b \circ c)$ . A Frobenius manifold is a manifold with a smooth structure of a Frobenius algebra on the tangent space at any point with certain compatibility conditions. Globally, we require the metric  $\Pi$  to be flat and the identity vector field  $e$  is constant with respect to its Levi-Civita connection. In this article, we show that orbits spaces of some non-reflection representations of finite groups acquire Frobenius manifold structures.

We use the following notations and facts for a finite group  $G$  and a linear representation  $\psi : G \rightarrow GL(V)$ , where  $V$  is a complex vector space. We denote by  $\mathbb{C}[V]$  the ring of polynomial functions on  $V$ ,  $\mathbb{C}[\psi]$  the subring of invariant polynomials in  $\mathbb{C}[V]$ , and  $\mathcal{O}(\psi)$  the orbits space of the action of  $G$  on  $V$ . Then  $\mathbb{C}[\psi]$  is finitely generated by homogeneous polynomials and  $\mathcal{O}(\psi)$  is a variety whose coordinate ring is  $\mathbb{C}[\psi]$  ([17], [2]). By Chevalley–Shephard–Todd theorem,  $\mathbb{C}[\psi]$  is a polynomial ring if and only if  $\psi$  is generated by pseudo-reflections. Let  $(x^1, \dots, x^n)$  be linear coordinates on  $V$  and  $f \in \mathbb{C}[\psi]$ . Then the Hessian  $H(f) := \frac{\partial^2 f}{\partial x^i \partial x^j}$  defines a bilinear form on the tangent space of  $\mathcal{O}(\psi)$  and if  $\det(H(f)) \neq 0$  then  $f$  is a minimal degree invariant polynomial ([18], page 6). In this article, we will drop the word pseudo as all representations will be representations over complex vector spaces.

Let  $\mathcal{W}$  be a finite irreducible Coxeter group or Shephard group of rank  $r$  and  $\rho_{ref}$  is the standard reflection representation of  $\mathcal{W}$ . Boris Dubrovin proved that the orbits space  $\mathcal{O}(\rho_{ref})$  acquires a polynomial Frobenius manifold structure ([4],[8]). This result led to the classification of irreducible semisimple polynomial Frobenius manifolds with positive degrees (see section 4.1 for more details). His method was used in [25] when  $\mathcal{W}$  is a Coxeter group of type  $B_r$  or  $D_r$  to construct  $r$  Frobenius manifolds on  $\mathcal{O}(\rho_{ref})$ . In this article, we show that linear representations of finite groups are a valuable source to construct examples of Frobenius manifolds even if the representations are not reflection representations.

We mention that Dubrovin and his collaborators constructed Frobenius manifolds using invariant rings of infinite discrete groups being extensions of affine Weyl groups ([7], [10], [26]). However, we focus in this article on linear representations of finite groups.

Let us fix a finite group  $G$  and a linear representation  $\psi : G \rightarrow GL(V)$  of rank  $r$ . Then we summarize **Dubrovin’s method** to construct Frobenius manifold structure on  $\mathcal{O}(\psi)$  as follows:

1. Fix homogeneous invariants polynomial  $f_1$  of the minimal degree.
2. Verify that the inverse of the Hessian  $H(f_1)$  defines a contravariant flat metric  $\Omega_2$  on some open subset  $U$  of  $\mathcal{O}(\psi)$ . For example, this happens if  $\psi$  is a real representation (in this case degree  $f_1$  equals 2) [11] or  $\psi$  is the standard reflection representation of a Shephard group [18].
3. Construct another contravariant metric  $\Omega_1$  which forms with  $\Omega_2$  a regular quasihomogenous flat pencil of metrics (regular QFPM) on  $U$  (see section 2.2 for details).
4. Then using a theorem due to Dubrovin (see Theorem 2.6 below), we get a Frobenius manifold structure on  $U$  which depends on the representation  $\psi$  of  $G$  or  $\mathbb{C}[\psi]$ .

**Definition 1.1.** *By abuse of language, a Frobenius manifold structure obtained using Dubrovin’s method will be called a natural Frobenius manifold structure on the orbits space.*

Note that for a fixed metric  $\Omega_2$ , the problem of finding another metric  $\Omega_1$  such that  $(\Omega_2, \Omega_1)$  form a flat pencil of metric is not straightforward. For example, see the discussion on the classification of flat pencils of metrics related to the theory of Frobenius manifolds given in [9]. We also observe that an orbits space can have several natural Frobenius manifold structures. In this article, we will prove the orbits spaces of the following representations posses natural Frobenius manifold structures:

1. The standard reflection representation of a finite irreducible Coxeter group: We prove there is a natural rational Frobenius manifold structure different from the ones constructed in [4] and [25]. We give details in section 4.1.
2. The non-standard irreducible representation of dimension  $r$  of a Coxeter group of type  $A_r$ : We show that it is a non-reflection representation and we construct certain  $r$  algebraically independent invariant polynomials. Then, we show that the orbits space carries natural rational Frobenius manifold structures. We give the details in section 4.2.
3. Irreducible representations of dihedral groups and dicyclic groups: These groups have only rank 1 and 2 irreducible representations. We will prove that any rank 2 representation acquires two natural Frobenius manifold structures. See section 5 for details.
4. All finite subgroups of the special linear group  $SL_2(\mathbb{C})$ : We get natural polynomial and rational Frobenius manifold structures related to representations of the dihedral groups. See section 5.3.
5. All finite subgroups of the special linear group  $SL_3(\mathbb{C})$  where the invariant rings are complete intersection: Dubrovin's method fails on some of them and we find natural trivial Frobenius manifold structures on others. We give details in section 6.

As a consequence of this work, we noticed that Frobenius manifold structures on orbits spaces of some non-reflection representations appear in pairs. Analyzing such pairs led us to the notion of the conjugate Frobenius manifold structures and we wrote the details on a separated article [1]. We review this notion in section 3 and we show that the conjugate of a natural Frobenius manifold structure is a natural Frobenius manifold structure.

To make the article as self-contained as possible, we review in section 2.1 and 2.2 the definition of Frobenius manifold and its relation with flat pencils of metrics.

## 2 Flat pencil of metrics and Frobenius manifolds

We review in this section the relation between Frobenius manifolds and flat pencil of metrics.

### 2.1 Frobenius manifolds

Let  $M$  be a Frobenius manifold with flat metric  $\Pi$  and identity vector field  $e$ . In flat coordinates  $(t^1, \dots, t^r)$  for  $\Pi$  where  $e = \partial_{t^r}$  the compatibility conditions imply that there exists a function  $\mathbb{F}(t^1, \dots, t^r)$  which encodes the Frobenius structure, i.e., the flat metric is given by

$$\Pi_{ij}(t) = \Pi(\partial_{t^i}, \partial_{t^j}) = \partial_{t^r} \partial_{t^i} \partial_{t^j} \mathbb{F}(t) \quad (2.1)$$

and, setting  $\Omega_1(t)$  to be the inverse of the matrix  $\Pi(t)$ , the structure constants of the Frobenius algebra are given by

$$C_{ij}^k(t) = \Omega_1^{kp}(t) \partial_{t^p} \partial_{t^i} \partial_{t^j} \mathbb{F}(t).$$

Here, and in what follows, summation with respect to repeated upper and lower indices is assumed. In this article, we assume the quasihomogeneity condition for  $\mathbb{F}(t)$  takes the form

$$E\mathbb{F}(t) = d_i t^i \partial_{t^i} \mathbb{F}(t) = (3 - d) \mathbb{F}(t); \quad d_r = 1. \quad (2.2)$$

The vector field  $E = d_i t^i \partial_{t^i}$  is known as Euler vector field and it defines the degrees  $d_i$  and the charge  $d$  of  $M$ . The associativity of the Frobenius algebra implies that the potential  $\mathbb{F}(t)$  satisfies WDVV equations, i.e.,

$$\partial_{t^i} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \Omega_1^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^n} \mathbb{F}(t) = \partial_{t^n} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \Omega_1^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^i} \mathbb{F}(t), \quad \forall i, j, q, n. \quad (2.3)$$

We say  $M$  is a polynomial (resp. rational) if  $\mathbb{F}(t)$  is a polynomial (resp. rational) function.

**Definition 2.1.** Let  $M$  and  $\widetilde{M}$  be two Frobenius manifolds with flat metrics  $\Pi$  and  $\widetilde{\Pi}$ . Let  $\mathbb{F}$  and  $\widetilde{\mathbb{F}}$  be the corresponding potentials, respectively. We say  $M$  and  $\widetilde{M}$  are (locally) equivalent if there are open sets  $U \subseteq M$  and  $\widetilde{U} \subseteq \widetilde{M}$  with a local diffeomorphism  $\phi: U \rightarrow \widetilde{U}$  such that

$$\phi^* \widetilde{\Pi} = c^2 \Pi, \quad (2.4)$$

for some nonzero constant  $c$ , and  $\phi_*: T_t U \rightarrow T_{\phi(t)} \widetilde{U}$  is an isomorphism of Frobenius algebras.

Note that, if  $M$  and  $\widetilde{M}$  are equivalent Frobenius structures then it is not necessary that  $\phi^* \widetilde{\mathbb{F}} = \mathbb{F}$  [4].

## 2.2 Flat pencil of metrics

We review the relation between flat pencils of metrics and Frobenius manifolds outlined in [6].

Let  $M$  be a smooth manifold of dimension  $r$  and fix local coordinates  $(u^1, \dots, u^r)$  on  $M$ .

**Definition 2.2.** A symmetric bilinear form  $(\cdot, \cdot)$  on  $T^*M$  is called a contravariant metric if it is invertible on an open dense subset  $M_0 \subseteq M$ . We define the contravariant Christoffel symbols  $\Gamma_k^{ij}$  for a contravariant metric  $(\cdot, \cdot)$  by

$$\Gamma_k^{ij} := -\Omega^{im} \Gamma_{mk}^j$$

where  $\Gamma_{mk}^j$  are the Christoffel symbols of the metric  $\langle \cdot, \cdot \rangle$  defined on  $TM_0$  by the inverse of the matrix  $\Omega^{ij}(u) = (du^i, du^j)$ . We say the metric  $(\cdot, \cdot)$  is flat if  $\langle \cdot, \cdot \rangle$  is flat.

Let  $(\cdot, \cdot)$  be a contravariant metric on  $M$  and set  $\Omega^{ij}(u) = (du^i, du^j)$ . Then we will use  $\Omega$  to refer to the metric and  $\Omega(u)$  to refer to its matrix in the coordinates. In particular, the Lie derivative of  $(\cdot, \cdot)$  along a vector field  $X$  will be written  $\text{Lie}_X \Omega$  while  $X \Omega^{ij}$  means the vector field  $X$  acting on the entry  $\Omega^{ij}$ . The Christoffel symbols given in Definition 2.2 determine for  $\Omega$  the contravariant (resp. covariant) derivative  $\nabla^i$  (resp.  $\nabla_i$ ) along the covector  $du^i$  (resp. the vector field  $\partial_{u^i}$ ). They are related by the identity  $\nabla^i = \Omega^{ij}(u) \nabla_j$ .

**Definition 2.3.** A flat pencil of metrics (FPM) on  $M$  is a pair  $(\Omega_2, \Omega_1)$  of two flat contravariant metrics  $\Omega_2$  and  $\Omega_1$  on  $M$  satisfying

1.  $\Omega_2 + \lambda \Omega_1$  defines a flat metric on  $T^*M$  for a generic constant  $\lambda$ ,
2. the Christoffel symbols of  $\Omega_2 + \lambda \Omega_1$  are  $\Gamma_{2k}^{ij} + \lambda \Gamma_{1k}^{ij}$ , where  $\Gamma_{2k}^{ij}$  and  $\Gamma_{1k}^{ij}$  are the Christoffel symbols of  $\Omega_2$  and  $\Omega_1$ , respectively.

**Definition 2.4.** A flat pencil of metrics  $(\Omega_2, \Omega_1)$  on  $M$  is called *quasihomogeneous flat pencil of metrics (QFPM)* of degree  $d$  if there exists a function  $\tau$  on  $M$  such that the vector fields  $E$  and  $e$  defined by

$$\begin{aligned} E &= \nabla_2 \tau, & E^i &= \Omega_2^{ij}(u) \partial_{u^j} \tau \\ e &= \nabla_1 \tau, & e^i &= \Omega_1^{ij}(u) \partial_{u^j} \tau \end{aligned} \quad (2.5)$$

satisfy

$$[e, E] = e, \quad \text{Lie}_E \Omega_2 = (d-1)\Omega_2, \quad \text{Lie}_e \Omega_2 = \Omega_1 \quad \text{and} \quad \text{Lie}_e \Omega_1 = 0. \quad (2.6)$$

Such a QFPM is **regular** if the  $(1,1)$ -tensor

$$R_i^j = \frac{d-1}{2} \delta_i^j + \nabla_{1i} E^j \quad (2.7)$$

is nondegenerate on  $M$ .

We will use the following source for FPM.

**Lemma 2.5.** [4] Let  $\Omega_2$  be a contravariant flat metric on  $M$ . Assume that in the coordinates  $(u^1, \dots, u^r)$ ,  $\Omega_2^{ij}(u)$  and  $\Gamma_{2k}^{ij}(u)$  depend almost linearly on  $u^r$ . Suppose that  $\Omega_1 := \text{Lie}_{\partial_{u^r}} \Omega_2 = \partial_{u^r} \Omega_2(u)$  is nondegenerate. Then  $(\Omega_2, \Omega_1)$  form a FPM. The Christoffel symbols of  $\Omega_1$  has the form  $\Gamma_{1k}^{ij}(u) = \partial_{u^r} \Gamma_{2k}^{ij}(u)$ .

If  $M$  is a Frobenius manifold then  $M$  has a QFPM of degree  $d$  but it does not necessarily satisfy the regularity condition (2.7) [6]. In the notations of section 2.1, the QFPM consists of the intersection form  $\Omega_2(t)$  and the flat metric  $\Omega_1(t)$  where

$$\Omega_2^{ij}(t) := (d-1 + d_i + d_j) \Omega_1^{i\alpha} \Omega_1^{j\beta} \partial_{t^\alpha} \partial_{t^\beta} \mathbb{F}. \quad (2.8)$$

Furthermore,  $\tau = \Pi_{i1} t^i$  and  $E$  with  $e$  are defined by (2.5) and satisfy equations (2.6). The converse is given by the following theorem

**Theorem 2.6.** [6] Let  $M$  be a manifold carrying a regular QFPM  $(\Omega_2, \Omega_1)$  of degree  $d$ . Then there exists a unique Frobenius manifold structure on  $M$  of charge  $d$  where  $(\Omega_2, \Omega_1)$  is the associated QFPM.

### 3 Conjugate Frobenius Manifold and Dubrovin's method

We begin this section with a theorem proved in [1] which leads to the notion of conjugate Frobenius manifold structure. Then we will prove that the conjugate natural Frobenius manifold structure constructed on an orbits spaces is also natural.

**Theorem 3.1.** [1] Let  $M$  be a Frobenius manifold with the Euler vector field  $E$  and the identity vector field  $e$ . Suppose the associated QFPM is regular of degree  $d$  with a function  $\tau$ . Assume that

$$e(\tau) = 0 \quad \text{and} \quad E(\tau) = (1-d)\tau. \quad (3.1)$$

Then we can construct another Frobenius manifold structure on  $M \setminus \{\tau = 0\}$  of degree  $2-d$ . Moreover, we can apply the same method to the new Frobenius manifold structure and it leads to the original Frobenius manifold structure.

For a fixed Frobenius manifold the new structure that can be obtained using Theorem 3.1 will be called the conjugate Frobenius manifold structure.

Let  $M$  be a Frobenius manifold of degree  $d$ . Let  $T = (\Omega_2, \Omega_1)$  be the associated QFPM with a function  $\tau$ , the Euler vector field  $E$  and the identity vector field  $e$ . Suppose it satisfies the hypothesis of Theorem 3.1. Then the QFPM associated to the conjugate Frobenius manifold structure has the form  $\tilde{T} := (\Omega_2, \tilde{\Omega}_1)$  where  $\tilde{\Omega}_1 := \text{Lie}_{\tilde{e}}\Omega_2$  and the vector field  $\tilde{e} := \tau^{\frac{2}{1-d}}e$  [1].

Let us adapt the notations of section 2.1 and assume  $\Pi_{ij} = \delta_{i+j}^{r+1}$ , i.e., the potential  $\mathbb{F}$  has the standard form

$$\mathbb{F}(t) = \frac{1}{2}(t^r)^2 t^1 + \frac{1}{2}t^r \sum_{i=2}^{r-1} t^i t^{r-i+1} + G(t^1, \dots, t^{r-1}). \quad (3.2)$$

Then we get the following consequence of Theorem 3.1.

**Theorem 3.2.** [1] *Let  $M$  be a Frobenius manifold with charge  $d \neq 1$ . Suppose in the flat coordinates  $(t^1, \dots, t^r)$ , the potential  $\mathbb{F}(t)$  has the standard form (3.2) and the quasihomogeneity condition takes the form (2.2) with  $d_i \neq \frac{d_1}{2}$  for every  $i$ . Then we can construct the conjugate Frobenius manifold structure on  $M \setminus \{t^1 = 0\}$ . Moreover, flat coordinates for the conjugate Frobenius manifold are*

$$s^1 = -t^1, \quad s^i = t^i (t^1)^{\frac{d_1 - 2d_i}{d_1}} \quad \text{for } 1 < i < r, \quad s^r = \frac{1}{2} \sum_{i=1}^r t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1} - 1}. \quad (3.3)$$

In addition, the corresponding potential equals the potential obtained by applying the inversion symmetry to  $\mathbb{F}(t)$  and it is given by

$$\tilde{\mathbb{F}}(s) = (t^1)^{\frac{-4}{d_1}} \left( \mathbb{F}(t^1, \dots, t^r) - \frac{1}{2} t^r \sum_{i=1}^r t^i t^{r-i+1} \right). \quad (3.4)$$

The degrees  $\tilde{d}_i$  and the charge  $\tilde{d}$  of the conjugate Frobenius manifold structure are given by

$$\tilde{d}_1 = -d_1, \quad \tilde{d}_r = 1, \quad \tilde{d}_i = d_i - d_1 \quad \text{for } 1 < i < r, \quad \tilde{d} = 2 - d. \quad (3.5)$$

See ([5], Appendix B) for details about inversion symmetry of solutions to WDVV equations. From the point of view of this article, Theorem 3.2 explains the appearance of pairs of natural Frobenius manifold structures on orbits space of some linear representations of finite groups.

**Theorem 3.3.** *Let  $M$  be the orbits space of a linear representation of a finite group. Assume  $M$  inherits a natural Frobenius manifold structure which has a conjugate Frobenius manifold structure. Then the conjugate Frobenius manifold structure on  $M$  is also natural.*

*Proof.* Let  $T = (\Omega_2, \Omega_1)$  be the associated QFPM of the Frobenius manifold structure on  $M$  which is obtained using Dubrovin's method. Then  $\Omega_2$  is defined using the Hessian of a minimal invariant polynomial  $f_1$ . The QFPM associated to the conjugate Frobenius manifold has the same intersection form  $\Omega_2$  and hence it constructed by Dubrovin's method.  $\square$

For convenience, we write in examples, indices of coordinates using subscripts instead of superscripts.

**Example 3.4.** *The potential*

$$\mathbb{F} = \frac{t_1^3}{6} - \frac{1}{2} t_2^2 t_1 + \frac{1}{2} t_2^2 t_3 + \frac{1}{2} t_1 t_3^2. \quad (3.6)$$

defines two inequivalent trivial Frobenius manifold structures, i.e., both have charge  $d = 0$  and Euler vector field  $E = \sum t_i \partial_{t_i}$ . Setting the identity vector field to be  $\tilde{e} = \partial_{t_1}$ ,  $\mathbb{F}$  defines a Frobenius manifold structure

$\widehat{T}_3$  whose associated regular QFPM does not satisfy condition (3.1), i.e., it does not have a conjugate structure. While fixing the identity vector field  $e = \partial_{t_3}$ , we get a Frobenius manifold structure  $T_3$  which has conjugate. The associated regular QFPM  $(\Omega_2, \Omega_1)$  has  $\Omega_1^{ij}(t) = \delta_3^{i+j}$  while

$$\Omega_2(t) = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 - t_1 & -t_2 \\ t_3 & -t_2 & t_1 \end{pmatrix}$$

Setting

$$s_1 = -t_1, \quad s_2 = \frac{t_2}{t_1}, \quad s_3 = \frac{t_2^2}{2t_1^3} + \frac{t_3}{t_1^2}$$

the conjugate QFPM has  $\widetilde{\Omega}_1^{ij}(s) = \delta_3^{i+j}$  and

$$\Omega_2(s) = \begin{pmatrix} -s_1 & 0 & s_3 \\ 0 & s_3 + \frac{3s_2^2}{2s_1} + \frac{1}{s_1} & -\frac{s_3^2}{s_1^2} - \frac{2s_2}{s_1^2} \\ s_3 & -\frac{s_3^2}{s_1^2} - \frac{2s_2}{s_1^2} & \frac{3s_2^4}{4s_1^3} + \frac{3s_2^2}{s_1^3} - \frac{1}{s_1^3} \end{pmatrix}$$

The potential of the conjugate Frobenius manifold structure reads

$$\widetilde{\mathbb{F}}(s) = \frac{-1}{6s_1} + \frac{s_2^2}{2s_1} + \frac{s_2^4}{8s_1} + \frac{1}{2}s_2^2s_3 + \frac{1}{2}s_1s_3^2. \quad (3.7)$$

Here  $\widetilde{E} = -s_1\partial_{s_1} + s_3\partial_{s_3}$  and  $\widetilde{E}\widetilde{\mathbb{F}} = \widetilde{\mathbb{F}}$ .

## 4 Coxeter groups

### 4.1 The standard reflection representation

In this section, we recall the standard reflection representations of irreducible finite Coxeter groups and review the construction of natural Frobenius manifolds on their orbits space. Then we classify those having conjugate Frobenius manifold structures. Note that the conjugate Frobenius manifold structures will be rational and they are not known to be related to invariant theory of finite groups.

We fix an irreducible finite Coxeter system  $(\mathcal{W}, S)$  of rank  $r$ , i.e.,

$$\mathcal{W} = \langle S \mid (ss')^{m(s,s')} = 1; \forall s, s' \in S \rangle, \quad r = |S|. \quad (4.1)$$

Let  $V$  be the formal vector space over  $\mathbb{C}$  with basis  $\{\alpha_s \mid s \in S\}$ . Then the standard reflection representation of  $\mathcal{W}$  is defined by

$$\begin{aligned} \rho_{ref} : \mathcal{W} &\rightarrow GL(V), \quad s \mapsto R_s, \quad s \in S. \\ R_s(v) &:= v - 2B(\alpha_s, v)\alpha_s, \quad v \in V, \quad B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')}. \end{aligned}$$

Here  $B$  is the standard positive-definite Hermitian form on  $V$  which is invariant under  $\rho_{ref}$ . By Chevalley–Shephard–Todd theorem, the invariant ring  $\mathbb{C}[\rho_{ref}]$  is a polynomial ring generated by  $r$  homogeneous polynomial. We fix generators  $u^1, \dots, u^r$  for  $\mathbb{C}[\rho_{ref}]$ . We assume  $\deg u^i = \eta_i$  and

$$2 = \eta_1 < \eta_2 \leq \eta_3 \leq \dots \leq \eta_{r+1} < \eta_r. \quad (4.2)$$



These degrees are uniquely determined by the group  $\mathcal{W}$  [14].

We assume  $u^1$  equals the quadratic form of  $B$ . Hence, the inverse of the Hessian of  $u^1$  defines a flat contravariant metric  $\Omega_2$  on  $\mathcal{O}(\rho_{ref})$ . It is easy to prove that  $\Omega_2(u)$  is almost linear in  $u^r$  by analysing the degrees of  $\Omega_2^{ij}(u)$ . We fix the vector field  $e = \partial_{u^r}$ . Note that changing the generators of  $\mathbb{C}[\rho_{ref}]$ ,  $e$  is uniquely defined up to a constant factor. Setting  $\Omega_1 := \text{Lie}_e \Omega_2$  Dubrovin proved that  $T := (\Omega_2, \Omega_1)$  is a regular QFPM of charge  $\frac{\eta_r - 2}{\eta_r}$  [6]. In this case,  $\tau = \frac{1}{\eta_r} u_1$  and the vector field  $E$  is given by  $E = \frac{1}{\eta_r} \sum_i \eta_i u^i \partial_{u^i}$ . This result initiated what we call Dubrovin's method. We observe that  $E$  is uniquely defined and does not depend on the choice of invariants  $u^i$ . Also, we mention that the flat metric  $\Omega_1$  was studied by K. Saito [20], [19] and his results was very important to the work [4]. We restate Dubrovin's theorem.

**Theorem 4.1.** ([4], [6]) *The FPM  $(\Omega_2, \Omega_1)$  defines a unique (up to equivalence) natural polynomial Frobenius manifold on  $\mathcal{O}(\rho_{ref})$  with degrees  $\frac{\eta_i}{\eta_r}$  and charge  $\frac{\eta_r - 2}{\eta_r}$ .*

The following theorem was conjectured by Dubrovin and proved by C. Hertling.

**Theorem 4.2.** [13] *Any irreducible massive polynomial Frobenius manifold with positive degrees is isomorphic to a polynomial Frobenius manifold constructed by Theorem 4.1 on the orbit space of the standard reflection representation of an irreducible finite Coxeter group.*

The following theorem grants the existence of another natural Frobenius manifold structure on  $\mathcal{O}(\rho_{ref})$ .

**Theorem 4.3.** *The polynomial Frobenius manifold constructed by Theorem 4.1 on the orbits space  $\mathcal{O}(\rho_{ref})$  has a conjugate Frobenius manifold structure. Thus, we get a rational natural Frobenius manifold structure on  $\mathcal{O}(\rho_{ref})$ .*

*Proof.* There exist invariant polynomials  $t^1, \dots, t^r$  which form flat coordinates and the potential has the form (3.2) [4]. From the structure of the degrees, we can and we will apply Theorem 3.2 to get a rational conjugate Frobenius manifold. The last statement is a consequence of Theorem 3.3.  $\square$

Let us assume  $\mathcal{W}$  is of type  $B_r$ . Then Dafeng Zuo obtained  $r$  Frobenius manifold structures on  $\mathcal{O}(\rho_{ref})$  by fixing certain generators  $z^1, \dots, z^r$  for  $\mathbb{C}[\rho_{ref}]$  [25]. Under these generators,  $\Omega_2(z)$  and its Christoffel symbols  $\Gamma_{2k}^{ij}(z)$  are almost linear in each  $z^k$ ,  $k = 1, 2, \dots, r$ . Then he proved that Lemma 2.5 can be applied and he constructed  $r$  rational Frobenius manifold structures using the flat pencils of metrics  $\widehat{T}_k := (\Omega_2, \text{Lie}_{\partial_{z^k}} \Omega_2)$ . He also proved that the same Frobenius manifold structures can be constructed when  $\mathcal{W}$  is of type  $D_r$ . Even it is not written explicitly in [25], We confirm that they are natural Frobenius manifold structures as each  $\widehat{T}_k$  is regular QFPM of degree  $1 - \frac{1}{k}$  with  $\tau = \frac{1}{4k} z^1$ . Here  $e = \partial_{z^k}$ . Thus, we can obtain these Frobenius manifolds directly using Theorem 2.6. Here the structure of Zuo's theorem

**Theorem 4.4.** [25] *There exists a unique natural Frobenius structure for each  $1 \leq k \leq r$  of charge  $d = 1 - \frac{1}{k}$  on the orbit space  $\mathcal{O}(\rho_{ref})$  when  $\mathcal{W}$  is of type  $B_r$  and  $D_r$  polynomial in  $t^1, t^2, \dots, t^r, \frac{1}{t^r}$  such that:*

1. *The identity vector field is  $e = \frac{\partial}{\partial z^k} = \frac{\partial}{\partial t^k}$ .*
2. *The Euler vector field is  $E = \sum_{i=1}^r d_i t^i \partial_{t^i}$ , where*

$$d_1 = \frac{1}{k}, \quad d_i = \frac{i}{k} \quad \text{for } 2 \leq i \leq k, \quad d_i = \frac{2k(r-i) + r}{2k(r-k)} \quad \text{for } k+1 \leq i \leq r.$$

3. *The associated QFPM is  $\widehat{T}_k$ .*



Note that when  $k = 1$ ,  $\widehat{T}_1$  does not satisfy condition (3.1). Thus the corresponding Frobenius manifold structure has no conjugate. For  $k > 1$ , we get the following theorem.

**Theorem 4.5.** *For  $k > 1$ , the natural Frobenius manifold structure corresponding to  $\widehat{T}_k$  constructed by Theorem 4.4 has a conjugate Frobenius manifold structure which is also natural.*

*Proof.* Similar to the proof of Theorem 4.3, we apply Theorem 3.2 and Theorem 3.3. □

Considering Theorem 4.2, let  $K$  be the type of  $\mathcal{W}$ , then we say a Frobenius manifold is of type  $K$  (rep. of type  $\widehat{K}$ ) if it is isomorphic to a natural polynomial Frobenius manifold (resp. a natural conjugate Frobenius manifold) constructed on  $\mathcal{O}(\rho_{ref})$  by Theorem 4.1 (resp. Theorem 4.3).

**Example 4.6.** *We list in Table 1 all Frobenius structures constructed on  $\mathcal{O}(\rho_{ref})$  when  $\mathcal{W}$  is of rank 3 using the above theorems. We borrow the potentials of Frobenius structures of type  $A_3$ ,  $B_3$  and  $H_3$  from [6]. From these potentials, we find Frobenius manifold structures of type  $\widetilde{A}_3$ ,  $\widetilde{B}_3$  and  $\widetilde{H}_3$  using the formula (3.4). Then applying Theorem 4.4 to a Coxeter group of type  $B_3$ , we get a Frobenius manifold of type  $B_3$  (resp.  $A_3$ ) when  $k = 3$  (resp.  $k = 2$ ). For  $k = 1$ , we get a rational Frobenius manifold  $B_3^1$  which has no conjugate.*

Notations	$\mathbb{F}(t_1, t_2, t_3)$	$d_1, d_2, d_3$	d
$A_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_1^2t_2^2 + \frac{1}{60}t_1^5$	$\frac{1}{2}, \frac{3}{4}, 1$	$\frac{1}{2}$
$\widetilde{A}_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{t_2^4}{8t_1} + \frac{t_2^2}{4t_1^2} - \frac{1}{60t_1^3}$	$-\frac{1}{2}, \frac{1}{4}, 1$	$\frac{3}{2}$
$B_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{6}t_2^2t_1^3 + \frac{1}{6}t_2^3t_1 + \frac{1}{210}t_1^7$	$\frac{1}{3}, \frac{2}{3}, 1$	$\frac{2}{3}$
$\widetilde{B}_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{t_2^4}{8t_1} + \frac{t_2^3}{6t_1^2} - \frac{t_2^2}{6t_1^3} - \frac{1}{210t_1^5}$	$-\frac{1}{3}, \frac{1}{3}, 1$	$\frac{4}{3}$
$H_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{20}t_2^2t_1^5 + \frac{1}{6}t_2^3t_1^2 + \frac{1}{3960}t_1^{11}$	$\frac{1}{5}, \frac{3}{5}, 1$	$\frac{4}{5}$
$\widetilde{H}_3$	$\frac{1}{2}t_3^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{t_2^4}{8t_1} - \frac{t_2^3}{6t_1^2} - \frac{t_2^2}{20t_1^3} - \frac{1}{3960t_1^5}$	$-\frac{1}{5}, \frac{2}{5}, 1$	$\frac{6}{5}$
$B_3^1$	$\frac{1}{2}t_3^4 + \frac{3}{2}t_1t_2t_3 + \frac{1}{8}t_1^3 + \frac{1}{16}\frac{t_2^3}{t_3}$	$1, \frac{3}{4}, \frac{5}{4}$	0

Table 1: Frobenius manifolds on orbits spaces of reflection groups of rank 3

## 4.2 Sign times reflection representation

We keep the notations of the last section and we assume  $\mathcal{W}$  is of type  $A_r$ . We study an irreducible representation  $\rho_{new}$  of  $\mathcal{W}$  which can be defined using the sign representation and the representation  $\rho_{ref}$ . The definition will enable us to construct  $r$  invariant polynomials of  $\rho_{new}$ . We will prove the invariant ring  $\mathbb{C}[\rho_{new}]$  is not a polynomial ring when  $r > 2$ . We recall that the degrees of a complete set of generators of  $\mathbb{C}[\rho_{ref}]$  are  $2, 3, \dots, r+1$ .

We consider the sign representation of  $\mathcal{W}$ ,  $\rho_{sign} : \mathcal{W} \rightarrow \mathbb{C}^*$  defined by sending each element  $s \in S$  to  $-1$ . Then we define the representation  $\rho_{new}$  of  $\mathcal{W}$  by

$$\rho_{new} : \mathcal{W} \rightarrow GL(\mathbb{C} \otimes V), \quad \rho_{new}(w) = \rho_{sign}(w) \otimes \rho_{ref}(w), \quad \forall w \in \mathcal{W}. \quad (4.3)$$

Note that  $\rho_{new}$  is a real representation of rank  $r$ . The following proposition proves that  $\rho_{new}$  is an irreducible representation.

**Proposition 4.7.** *The new representation  $\rho_{new}$  is an irreducible representation of  $\mathcal{W}$ . Moreover,  $\rho_{new}$  and  $\rho_{ref}$  are isomorphic when  $r = 2$  and different otherwise.*

*Proof.* Recall that if  $\chi_\psi$  denotes the character of a representation  $\psi$  of a finite group  $G$ , then  $\psi$  is irreducible if and only if [22]

$$\frac{1}{|G|} \sum_{g \in G} \chi_\psi(g) \overline{\chi_\psi(g)} = 1. \quad (4.4)$$

Note that  $\rho_{ref}$  and  $\rho_{sign}$  are irreducible representations and

$$\chi_{\rho_{new}}(w) = \chi_{\rho_{sign}}(w) \chi_{\rho_{ref}}(w).$$

Then

$$\begin{aligned} \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \chi_{\rho_{new}}(w) \overline{\chi_{\rho_{new}}(w)} &= \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} (\chi_{\rho_{sign}}(w) \chi_{\rho_{ref}}(w)) \overline{(\chi_{\rho_{sign}}(w) \chi_{\rho_{ref}}(w))} \\ &= \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} (\chi_{\rho_{ref}}(w) \overline{\chi_{\rho_{ref}}(w)}) = 1. \end{aligned} \quad (4.5)$$

For the second part, note that for any generator  $s \in S$ ,  $\chi_{\rho_{new}}(s) = -\chi_{\rho_{ref}}(s) = -(r-2)$ . Hence, the two representations are different when  $r \neq 2$ . For  $r = 2$ , we can check that  $\rho_{new}$  is equivalent to  $\rho_{ref}$  by direct computations. □

For the remainder of this section we assume the rank  $r > 2$ . Recall that the Coxeter group of type  $A_r$  is isomorphic to the symmetric group  $S_{r+1}$ . Thus, irreducible representations of  $A_r$  are in one to one correspondence with the partition of  $r+1$ . For a given partition  $\lambda$  of  $r+1$ , the corresponding irreducible representation can be constructed using Young tableaux associated to  $\lambda$  [11]. Under this construction, the reflection representation  $\rho_{ref}$  is associated with the partition  $[r, 1]$ ,  $\rho_{sign}$  is associated with the partition  $[r+1]$  while  $\rho_{new}$  is associated with  $[2, 1, 1, \dots, 1]$ . The character of each representation is given by Frobenius formula [11]. We use this formula to prove the following proposition.

**Proposition 4.8.** *The irreducible representation  $\rho_{new}$  is not a reflection representation. In particular the ring  $\mathbb{C}[\rho_{new}]$  is not a polynomial ring.*

*Proof.* Assume that  $\rho_{new}$  is a reflection representation. Then, it is generated by a set of involutions  $w_1, \dots, w_r$ . Since  $\rho_{new}$  is a real representation, we must have  $\chi_{\rho_{new}}(w_i) = r-2$ . From  $\rho_{new}(w_i) = \rho_{sign}(w_i) \rho_{ref}(w_i)$ , we have  $\rho_{sign}(w_i) = -1$ , since if  $\rho_{sign}(w_i) = 1$ , then  $\rho_{ref}(w_i)$  is a reflection and we get a contradiction. Thus,  $\chi_{\rho_{ref}}(w_i) = 2-r$ . In the one-to-one correspondence between conjugacy classes of  $S_{r+1}$  and partitions of  $r+1$ ,  $w_i$  corresponds to a partition of the form  $[2, 2, \dots, 2, 1, 1, \dots, 1] = [2^p, 1^q]$  with  $2p+q = r+1$ ,  $p > 0$ . Using Frobenius formula,  $\chi_{\rho_{ref}}(w_i)$  equals the coefficient of  $x^{r+1}y$  in the expansion  $(x-y)(x^2+y^2)^p(x+y)^{r+1-2p}$ . Hence,  $\chi_{\rho_{ref}}(w_i) = r-2p$ . Using the fact that  $2p \leq r+1$  and  $\chi_{\rho_{ref}}(w_i) = 2-r$  we get  $r \leq 3$ . However, the case  $r = 3$  is excluded by direct computations. □

We study the ring  $\mathbb{C}[\rho_{new}]$  in order to use Dubrovin's method. We fix a basis  $e_1, e_2, \dots, e_r$  for  $V$  and let  $x^1, \dots, x^r$  be the dual basis satisfying  $x^i(e_j) = \delta_j^i$ . Then  $\tilde{e}_i := \mathbf{1} \otimes e_i$ ,  $i = 1, \dots, r$  form a basis of  $\mathbb{C} \otimes V$  and we get a natural isomorphism

$$\theta : \mathbb{C} \otimes V \rightarrow V, \tilde{e}_i \mapsto e_i. \quad (4.6)$$

Then the pullback  $\tilde{x}^i = \theta^*(x^i)$  defines the dual basis of  $\tilde{e}_i$ . Let  $w \in \mathcal{W}$  and  $a_i^j$  be the matrix of  $\rho_{ref}(w)$  under the basis  $e_i$ . Then  $\rho_{new}(w)(\tilde{e}_i) = \rho_{sign}(w) \mathbf{1} \otimes \rho_{ref}(w)e_i = \rho_{sign}(w) a_i^j \tilde{e}_j$ . Therefore,  $\rho_{new}(w) = \rho_{sign}(w) \rho_{ref}(w)$ .

**Lemma 4.9.** *Let  $w \in \mathcal{W}$  with  $\rho_{\text{sign}}(w)\rho_{\text{new}}(w) \notin \rho_{\text{ref}}(\mathcal{W})$  and  $f \in \mathbb{C}[\rho_{\text{ref}}]$  be homogeneous polynomial. Then*

$$w \cdot \theta^*(f) = (\rho_{\text{sign}}(w))^{\deg(f)} \theta^*(f). \quad (4.7)$$

*In particular, if degree  $f$  is even then  $\theta^*(f) \in \mathbb{C}[\rho_{\text{new}}]$ .*

*Proof.* We obtain  $\theta^*(f)$  simply by replacing the coordinate  $x^i$  with  $\tilde{x}^i$ . Therefore,

$$\begin{aligned} w \cdot \theta^*(f)(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) &= \theta^*(f)(\rho_{\text{new}}(w)\tilde{x}^1, \rho_{\text{new}}(w)\tilde{x}^2, \dots, \rho_{\text{new}}(w)\tilde{x}^n) \\ &= \theta^*(f)\left(\rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\tilde{x}^1, \rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\tilde{x}^2, \dots, \right. \\ &\quad \left. \rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\tilde{x}^n\right) \\ &= (\rho_{\text{sign}}(w))^{\deg(f)} \theta^*(f)(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n). \end{aligned}$$

□

Let  $z^1, \dots, z^r$  be algebraically independent invariant polynomials of  $\rho_{\text{new}}$  and  $u^1, \dots, u^r$  be the generators of  $\mathbb{C}[\rho_{\text{ref}}]$  (in the notation of section 4.1). We assume  $z^1 = \theta^*(u^1)$ . Hence, the Hessian of  $z^1$  defines a contravariant flat metric  $\Omega_2$  on  $\mathcal{O}(\rho_{\text{new}})$ . Examples show that the entries of  $\Omega_2(z)$  are rational in general and it is hard to construct flat pencil of metrics. We overcome this problem by defining certain invariants for  $\rho_{\text{new}}$  which also leads to the construction of Frobenius manifold structures.

**Proposition 4.10.** *There exist  $r$  algebraically independent invariant polynomials  $z^1, z^2, \dots, z^r$  of  $\rho_{\text{new}}$  with the degrees*

$$2, 4, 6, \dots, 2\lfloor \frac{r+1}{2} \rfloor; 6, 8, \dots, 2\lceil \frac{r+3}{2} \rceil. \quad (4.8)$$

*Proof.* We will use the invariants  $u^1, \dots, u^r$  of  $\rho_{\text{ref}}$  to construct invariants of  $\rho_{\text{new}}$ . We set  $I = \{i : \eta_i \text{ is even}\}$  and  $J = \{j : \eta_j \text{ is odd}\}$ . Using Lemma 4.9,  $\theta^*(u^i)$  is an invariant of  $\rho_{\text{new}}$  for any  $i \in I$ . Let  $\kappa$  be the minimal index in  $J$ . Then  $\theta^*(u^\kappa u^j)$  is an invariant of  $\rho_{\text{new}}$  for any  $j \in J$ . By this way, we construct  $r$  invariants polynomial,  $z^1, \dots, z^r$  for  $\rho_{\text{new}}$  with the degrees given in (4.8). Note that any polynomial in  $z^1, \dots, z^r$  can be written as a polynomial in  $u^1, \dots, u^r$ . Hence,  $z^1, \dots, z^r$  are algebraically independent. □

**Remark 4.11.** *We observe that the invariant polynomials constructed by Proposition 4.10 do not necessarily form a set of primary invariant polynomials of  $\rho_{\text{new}}$ . According to the invariant theory [2], the product of the degrees of primary invariants is divisible by the order of the group. For example, when  $\mathcal{W}$  is type  $A_4$ , the degrees of  $z^i$  are 2, 4, 6, 8. The product of these degrees is not divisible by the order 120 of the group.*

We keep the notations  $z^1, \dots, z^r$  for the invariant polynomials of  $\rho_{\text{new}}$  constructed in Proposition 4.10.

**Theorem 4.12.** *The orbits space  $\mathcal{O}(\rho_{\text{new}})$  has natural Frobenius manifold structures isomorphic to the natural Frobenius manifolds structures defined on  $\mathcal{O}(\rho_{\text{ref}})$  by Theorem 4.1 and Theorem 4.2.*

*Proof.* We consider the map  $(u^1, \dots, u^r) \rightarrow (z^1, z^2, \dots, z^r)$  given in Proposition 4.10 as diffeomorphism on some open subset of  $u^\kappa \neq 0$  where  $\kappa$  is defined in the proof of Proposition 4.10. Note that, under this diffeomorphism, the metric defined by the Hessian of  $u^1$  is identified with the metric defined by the Hessian of  $z^1$ . Thus, we can transfer to  $\mathcal{O}(\rho_{\text{new}})$ , any regular QFPM given by the Theorems 4.1 and 4.2. In this way, we obtain natural Frobenius manifold structures on  $\mathcal{O}(\rho_{\text{new}})$ . □

**Example 4.13.** The irreducible reflection representation  $\rho_{ref}$  of Coxeter group of type  $A_4$  is generated by the matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (4.9)$$

The polynomial ring  $\mathbb{C}[\rho_{ref}] = \mathbb{C}[u_1, u_2, u_3, u_4]$  where

$$\begin{aligned} u_1 &= x_1^2 - \frac{1}{2}x_1x_2 - \frac{1}{2}x_1x_3 - \frac{1}{2}x_1x_4 + x_2^2 - \frac{1}{2}x_2x_3 - \frac{1}{2}x_2x_4 + x_3^2 - \frac{1}{2}x_3x_4 + x_4^2, \\ u_2 &= x_1^3 - \frac{3}{4}x_1^2x_2 - \frac{3}{4}x_1^2x_3 - \frac{3}{4}x_1^2x_4 - \frac{3}{4}x_1x_2^2 + x_1x_2x_3 + x_1x_2x_4 - \frac{3}{4}x_1x_3^2 + x_1x_3x_4 - \frac{3}{4}x_1x_4^2 + x_2^3 - \frac{3}{4}x_2^2x_3 \\ &\quad - \frac{3}{4}x_2^2x_4 - \frac{3}{4}x_2x_3^2 + x_2x_3x_4 - \frac{3}{4}x_2x_4^2 + x_3^3 - \frac{3}{4}x_3^2x_4 - \frac{3}{4}x_3x_4^2 + x_4^3, \\ u_3 &= x_1^4 - x_1^3x_2 - x_1^3x_3 - x_1^3x_4 + x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 - x_1x_2^3 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_2x_3^2 - 3x_1x_2x_3x_4 \\ &\quad + x_1x_2x_4^2 - x_1x_3^3 + x_1x_3^2x_4 + x_1x_3x_4^2 - x_1x_4^3 + x_2^4 - x_2^3x_3 - x_2^3x_4 + x_2^2x_3x_4 - x_2x_3^3 + x_2x_3^2x_4 + x_2x_3x_4^2 \\ &\quad - x_2x_4^3 + x_3^4 - x_3^3x_4 - x_3x_4^3 + x_4^4, \\ u_4 &= x_1^5 - \frac{5}{4}x_1^4x_2 - \frac{5}{4}x_1^4x_3 - \frac{5}{4}x_1^4x_4 + \frac{5}{3}x_1^3x_2x_3 + \frac{5}{3}x_1^3x_2x_4 + \frac{5}{3}x_1^3x_3x_4 - \frac{5}{2}x_1^2x_2x_3x_4 - \frac{5}{4}x_1x_2^4 + \frac{5}{3}x_1x_2^3x_3 \\ &\quad + \frac{5}{3}x_1x_2^3x_4 - \frac{5}{2}x_1x_2^2x_3x_4 + \frac{5}{3}x_1x_2^2x_3^2 - \frac{5}{2}x_1x_2^2x_3x_4 - \frac{5}{2}x_1x_2x_3^2x_4 + \frac{5}{3}x_1x_2x_4^3 - \frac{5}{4}x_1x_3^4 + \frac{5}{3}x_1x_3^3x_4 \\ &\quad + \frac{5}{3}x_1x_3x_4^3 - \frac{5}{4}x_1x_4^4 + x_2^5 - \frac{5}{4}x_2^4x_3 - \frac{5}{4}x_2^4x_4 + \frac{5}{3}x_2^3x_3x_4 - \frac{5}{4}x_2^3x_4^2 + \frac{5}{3}x_2^2x_3^2x_4 + \frac{5}{3}x_2^2x_3x_4^2 - \frac{5}{4}x_2^2x_4^3 + x_3^5 \\ &\quad - \frac{5}{4}x_3^4x_4 - \frac{5}{4}x_3x_4^4 + x_4^5. \end{aligned}$$

The Frobenius manifold of type  $A_4$  is a result of the regular QFPM consists of  $\Omega_2(u)$  and  $\Omega_1 = \partial_{u_4}\Omega_2(u)$  where  $\Omega_2(u)$  is defined by the Hessian of  $u_1$ . The representation  $\rho_{new}$  is generated by  $\tau$  and  $-\sigma$ . Then the primary invariants of  $\rho_{new}$  have degrees 2, 4, 6, 10 while the secondary invariants have degrees 8, 13, 15. The Hessian of the degree 2 invariant  $z_1$  leads to the flat contravariant metric  $\Omega_2(z)$  but it is hard to find a FPM. We fix the following 4 invariants polynomials for  $\mathcal{O}(\rho_{new})$  of degrees 2, 4, 6 and 8:

$$z_1 = u_1, \quad z_2 = u_3, \quad z_3 = u_2^2, \quad z_4 = u_2u_4.$$

Then the matrix of  $\Omega_2(z)$  consists of the columns

$$\begin{aligned} \Omega_2^{i1}(z) &= \begin{pmatrix} z_1 \\ 2z_2 \\ 3z_3 \\ 4z_4 \end{pmatrix}, \quad \Omega_2^{i2}(z) = \begin{pmatrix} 2z_2 \\ -\frac{64}{625}z_1^3 + \frac{68}{25}z_1z_2 + \frac{864}{625}z_3 \\ \frac{12}{5}z_1z_3 + \frac{18}{5}z_4 \\ \frac{64}{75}z_1^2z_3 + \frac{43}{15}z_2z_3 + \frac{62}{25}z_1z_4 + \frac{9}{5}z_4^2 \end{pmatrix}, \\ \Omega_2^{i3}(z) &= \begin{pmatrix} 3z_3 \\ \frac{12}{5}z_1z_3 + \frac{18}{5}z_4 \\ \frac{2}{3}z_1^2z_3 + \frac{25}{3}z_2z_3 \\ -\frac{26}{45}z_1^3z_3 + \frac{95}{18}z_1z_2z_3 + \frac{14}{5}z_3^2 + \frac{1}{3}z_1^2z_4 + \frac{25}{6}z_2z_4 \end{pmatrix} \end{aligned}$$

and

$$\Omega_2^{i4}(z) = \begin{pmatrix} 4z_4 \\ \frac{64}{75}z_1^2z_3 + \frac{43}{15}z_2z_3 + \frac{62}{25}z_1z_4 + \frac{9}{5}z_4^2 \\ -\frac{26}{45}z_1^3z_3 + \frac{95}{18}z_1z_2z_3 + \frac{14}{5}z_3^2 + \frac{1}{3}z_1^2z_4 + \frac{25}{6}z_2z_4 \\ \frac{214}{2025}z_1^4z_3 + \frac{52}{81}z_1^2z_2z_3 + \frac{625}{324}z_2^2z_3 + \frac{56}{25}z_1z_3^2 - \frac{26}{45}z_1^3z_4 + \frac{95}{18}z_1z_2z_4 + \frac{62}{15}z_3z_4 + \frac{1}{z_3}z_1^2z_4^2 + \frac{25}{12}z_2z_4^2 \end{pmatrix}$$

Therefore, on  $\mathcal{O}(\rho_{\text{new}})$ , we get the regular QFPM formed by  $\Omega_2(z)$  and  $\Omega_1(z) = \text{Lie}_e \Omega_2$  where  $e = \sqrt{z_3} \partial_{z_4}$ . Of course, the resulted Frobenius manifold is of type  $A_4$ .

**Remark 4.14.** It is straightforward to generalize the results of this section to other types of Coxeter groups and we obtain natural Frobenius manifolds on  $\mathcal{O}(\rho_{\text{new}})$ . But we lack sorting out when  $\rho_{\text{new}}$  is not a reflections group (i.e. see Proposition 4.8). Robert Howett informed us that when  $\mathcal{W}$  is of type  $E_8$ , the representation  $\rho_{\text{new}}$  is generated by reflections.

## 5 Dihedral and dicyclic groups

In this section, we give results of applying Dubrovin's method to irreducible representations of the dihedral groups (Coxeter groups of type)  $I_2(m)$ ,  $m > 2$  and Dicyclic groups  $\text{Dic}_m$ . We mention that Dubrovin computed by an ad-hoc procedure all possible potentials of 2-dimensional Frobenius manifolds [5]. Here we find some of them are related to invariant theory of finite groups.

### 5.1 Dihedral groups

Irreducible representations of  $I_2(m)$  are of rank 1 or 2. Let  $\xi_m$  be a primitive  $m$ -th root of unity. The rank 2 representations are  $\rho_k$ ,  $k = 1, 2, \dots, \frac{m-2}{2}$  for even  $m$ , and  $k = 1, 2, \dots, \frac{m-1}{2}$  for odd  $m$ . Here,  $\rho_k$  is generated by the matrices

$$\begin{pmatrix} \xi_m^k & 0 \\ 0 & \xi_m^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.1)$$

When  $k = 1$ , we get the standard reflection representation of  $I_2(m)$ . We observe that  $\mathbb{C}[\rho_k]$  can be interpreted as the invariant ring of the standard reflection representation of  $I_2(h)$  where  $h = \frac{m}{\gcd(m,k)}$ , i.e. it is generated by

$$t_1 = \frac{1}{h} x_1 x_2, \quad t_2 = x_1^h + x_2^h.$$

Hence, applying Dubrovin's method, we get the polynomial Frobenius manifold of type  $I_2(h)$  and its conjugate  $\tilde{I}_2(h)$  obtained by Theorem 3.2.

### 5.2 Dicyclic groups

We fix a natural number  $m$ . The dicyclic group  $\text{Dic}_m$  is a group of order  $4m$  defined by

$$\text{Dic}_m = \langle \sigma, \alpha | \sigma^{2m} = 1, \alpha^2 = \sigma^m, \alpha^{-1} \sigma \alpha = \sigma^{-1} \rangle. \quad (5.2)$$

The irreducible representation of  $\text{Dic}_m$  are of rank 1 or 2. The 2-dimensional irreducible representations  $\psi_k$  and  $\varrho_l$  are defined by setting

$$\psi_k(\sigma) = \begin{pmatrix} \xi_{2m}^k & 0 \\ 0 & \xi_{2m}^{-k} \end{pmatrix}, \quad \psi_k(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5.3)$$

and

$$\varrho_l(\sigma) = \begin{pmatrix} \xi_m^k & 0 \\ 0 & \xi_m^{-k} \end{pmatrix}, \quad \varrho_l(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.4)$$

Here  $1 \leq k \leq \frac{m-2}{2}$  and  $1 \leq l \leq m-1$  when  $m$  is even while  $1 \leq k \leq \frac{m-1}{2}$  and  $1 \leq l \leq m-2$  when  $m$  is odd. Note that  $\psi_1$  is the standard representation of  $\text{Dic}_m$  in the literature. We observe that the invariant ring  $\mathbb{C}[\varrho_l]$  can be interpreted as the invariant ring  $\mathbb{C}[\rho_l]$  where  $\rho_l$  is the representation of  $I_2(m)$  given in

section 5.1. Thus, the result of applying Dubrovin's method to  $\varrho_l$  is given in that section. We consider here the representations  $\psi_k$ . Let us fix the integer  $k$  and set  $h = \frac{m}{\gcd(m,k)}$ . We define

$$u_1 = x_1^2 x_2^2, \quad u_2 = x_1^{2h} + x_2^{2h}, \quad u_3 = x_1 x_2 (x_1^{2h} - x_2^{2h}). \quad (5.5)$$

It is straightforward to verify that  $u_1, u_2$  and  $u_3$  are invariants under the action of  $\psi_k$ .

**Proposition 5.1.** *The invariant ring  $\mathbb{C}[\psi_k]$  is generated by  $u_1, u_2$  and  $u_3$ .*

*Proof.* A general homogeneous polynomial of degree  $q$  has the form

$$f(x_1, x_2) = a_q x_1^q + a_{q-1} x_1^{q-1} x_2 + \cdots + a_1 x_1 x_2^{q-1} + a_0 x_2^q \quad (5.6)$$

where  $a_0, \dots, a_q \in \mathbb{C}$ . Being invariant under  $\psi_k(\alpha)$ , we get

$$\begin{aligned} f &= a_q x_1^q + a_{q-1} x_1^{q-1} x_2 + \cdots + a_1 x_1 x_2^{q-1} + a_0 x_2^q \\ &= \psi_k(\alpha) f = (-1)^q a_q x_2^q + (-1)^{q-1} a_{q-1} x_2^{q-1} x_1 + \cdots - a_1 x_2 x_1^{q-1} + a_0 x_1^q \end{aligned}$$

Thus  $a_i = (-1)^{q-i} a_{q-i}$  for all  $i = 0, \dots, q$ . Similarly, the invariant of  $f$  under  $\psi_k(\alpha^2)$  implies  $q$  is even. Hence,  $f$  has the form

$$f = \sum_{i=0}^{\frac{q}{2}} a_{q-i} (x_1 x_2)^i [x_1^{q-2i} + (-1)^{q-i} x_2^{q-2i}].$$

Moreover,

$$f = \psi_k(\sigma) f = \sum_{i=0}^{\frac{q}{2}} a_{q-i} (x_1 x_2)^i [\xi^{-k(q-2i)} x_1^{q-2i} + (-1)^{q-i} \xi^{k(q-2i)} x_2^{q-2i}]$$

implies  $k(q-2i) \equiv 0 \pmod{2m}$ . Then  $q-2i = 2hl$  for some integer  $l$ . Therefore, we can write

$$f = \sum_{q=2hl+2i} a_{q-i} (x_1 x_2)^i [x_1^{2hl} + (-1)^{q-i} x_2^{2hl}]. \quad (5.7)$$

Now we show that  $f \in \mathbb{F}[u_1, u_2, u_3]$ . It is sufficient to prove  $\tilde{f}_l = x_1^{2hl} + x_2^{2hl}$  and  $\widehat{f}_l = x_1 x_2 (x_1^{2hl} - x_2^{2hl})$  are invariant for every natural number  $l$ . When  $l = 1$ ,  $\tilde{f}_l = u_2$  and  $\widehat{f}_l = u_3$ . For  $l + 1$ , we get

$$\begin{aligned} \tilde{f}_{l+1} &= x_1^{2h(l+1)} + x_2^{2h(l+1)} = (x_1^{2h} + x_2^{2h})^{l+1} - \sum_{d=1}^l \binom{l+1}{d} x_1^{2hd} x_2^{(l+1-d)2h} \\ &= (x_1^{2h} + x_2^{2h})^{l+1} - \sum_{d=1}^{\lfloor \frac{l}{2} \rfloor} \binom{l+1}{d} (x_1 x_2)^{2hd} (x_2^{2h(l+1-2d)} + x_1^{2h(l+1-2d)}). \end{aligned} \quad (5.8)$$

Since  $d \geq 1$ , we have  $l+1-2d \leq l-1 < l$ . Therefore, by the induction  $\tilde{f}_{l+1} \in \mathbb{C}[u_1, u_2, u_3]$ . Likewise  $\widehat{f}_{l+1} \in \mathbb{C}[u_1, u_2, u_3]$  since

$$\begin{aligned} \widehat{f}_{l+1} &= x_1 x_2 (x_1^{2h} - x_2^{2h}) (x_1^{2hl} + x_1^{2h(l-1)} x_2^{2h} + x_1^{2h(l-2)} x_2^{4h} + \cdots + x_2^{2hl}) \\ &= x_1 x_2 (x_1^{2h} - x_2^{2h}) [(x_1^{2hl} + x_2^{2hl}) + (x_1 x_2)^{2h} (x_1^{2h(l-2)} + x_2^{2h(l-2)}) \\ &\quad + (x_1 x_2)^{4h} (x_1^{2h(l-4)} + x_2^{2h(l-4)}) + \cdots]. \end{aligned} \quad (5.9)$$

This proves the proposition.  $\square$

We note that the invariant ring  $\mathbb{C}[\psi_k]$  can be interpreted as the invariant ring of the standard representation of  $\text{Dic}_h$ . A result of applying Dubrovin's method is obtained in [3]. We summarize the construction here.

The flat contravariant metric defined by the inverse of the Hessian of  $u_1$  is

$$\Omega_2(u) = \begin{pmatrix} \frac{4}{3}u_1 & \frac{2h}{3}u_2 \\ \frac{2h}{3}u_2 & -\frac{2h^2}{3u_1}(u_2^2 - 6u_1^h) \end{pmatrix}. \quad (5.10)$$

Then we considered a vector field  $e$  in the form  $e = f(u_1)\partial_{u_2}$  and imposed the conditions  $\text{Lie}_e\Omega_2$  is flat and  $\text{Lie}_e^2\Omega_2 = 0$ . These conditions lead to two independent solutions

$$f_{\pm} = u_1^{\frac{h}{2}(1 \pm \sqrt{3})}. \quad (5.11)$$

Setting  $e_{\pm} = f_{\pm}\partial_{u_2} = u_1^{\frac{h}{2}(1 \pm \sqrt{3})}\partial_{u_2}$ , we get regular quasihomogenous flat pencils of metrics  $(\Omega_2, \text{Lie}_{e_{\pm}}\Omega_2)$  of degree  $d = \frac{\sqrt{3}h \pm 2}{\sqrt{3}h}$  with  $\tau = \mp \frac{\sqrt{3}}{2h}u_1$ . The resulting Frobenius manifold structures are conjugate to each other. The corresponding flat coordinates are

$$t_1 = \mp \frac{\sqrt{3}}{2h}u_1, \quad t_2 = u_2 u_1^{\frac{\mp h}{2}(\sqrt{3} \pm 1)} \quad (5.12)$$

Which lead to the potentials

$$\mathbb{F} = \frac{2^{\mp\sqrt{3}h} 3^{\frac{1}{2}(1 \pm \sqrt{3}h)} (ht_1)^{1 \mp \sqrt{3}h}}{\mp(3h^2 - 1)} + \frac{1}{2}t_1 t_2^2 \quad (5.13)$$

of the degrees  $\mp \frac{2}{\sqrt{3}h}$  and 1.

### 5.3 Finite subgroups of $SL_2(\mathbb{C})$

In this section we use Dubrovin's method on finite non trivial subgroups of  $SL_2(\mathbb{C})$ . They are classified up to conjugation and they are called binary polyhedral groups. They consist of the cyclic groups  $\mathcal{C}_m$  and binary dihedral groups  $\mathcal{D}_m$ , binary tetrahedral group  $\mathcal{T}$ , binary octahedral group  $\mathcal{O}$  and binary icosahedral group  $\mathcal{I}$ . We treat them as representations of the corresponding groups. It is known that the invariant rings of these representations are not polynomial rings and the relations between the generators lead to the classification of simple hypersurface singularities. We use the sets of generators of the invariant rings listed in [15]. Applying Dubrovin's method, we obtain natural polynomial Frobenius manifold structure and their conjugations (as given in section 4.1). We write below only the flat coordinates and the type of the resulting polynomial Frobenius manifold structures. Note that the findings are not apparent from examining the invariant rings.

1. Cyclic groups  $\mathcal{C}_m$ : Here  $m \geq 2$  and the invariant ring is generated by  $xy$ ,  $x^m$ ,  $y^m$ . We fix the following invariant polynomials

$$t_1 = \frac{1}{m}xy, \quad t_2 = x^m + y^m$$

Then the ring generated by  $t_1$  and  $t_2$  is isomorphic to the invariant ring of the standard representation of the dihedral group  $I_2(m)$ . Thus, using Dubrovin's method and  $(t_1, t_2)$  as coordinates on the orbits space, we get Frobenius manifold of type  $I_2(m)$ .

In case we set  $t_1 = \frac{1}{m}xy$  and  $t_2 = x^m$ , we get the WDVV solution  $\frac{1}{2}t_1 t_2^2$ . It corresponds to a trivial Frobenius manifold structure but here the natural charge is  $\frac{m-2}{m}$  while the degrees are  $\frac{1}{m}$  and 1.



2. The binary dihedral group  $\mathcal{D}_m$ : This is the standard representation of the dicyclic group  $\text{Dic}_m$ . A result of applying Dubrovin's method is given in section 5.2.
3. The binary tetrahedral  $\mathcal{T}$ : We fix the following set of generators of the invariant ring

$$\begin{aligned} t_1 &= \frac{5}{12}xy(x^4 - y^4), \quad t_2 = (x^4 + y^4)^3 - 36x^4y^4(x^4 + y^4). \\ t_3 &= 16x^4y^4 + 2(x^4 - y^4). \end{aligned} \quad (5.14)$$

We choose  $(t_1, t_2)$  as coordinates on the orbits space. Then the Hessian of  $\frac{12}{5}t_1$  defines a flat metric  $\Omega_2(t)$  linear in  $t_2$ . Here, we apply Lemma 2.5 and we get a regular QFPM of degree  $d = \frac{1}{2}$  with  $\tau = t_1$  consists of

$$\Omega_2^{ij}(t) = \begin{pmatrix} \frac{1}{2}t_1 & t_2 \\ t_2 & -\frac{4478976}{625}t_1^3 \end{pmatrix}, \quad \Omega_1^{ij} = \text{Lie}_{\partial_{t_2}} \Omega_2^{ij}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5.15)$$

The resulting Frobenius manifold is of type  $I_2(4)$ .

4. The binary octahedral  $\mathcal{O}$ : Let us fix the generators of the invariant ring to be

$$\begin{aligned} t_1 &= \frac{7}{12}(16x^4y^4 + (x^4 - y^4)^2), \quad t_2 = (xy(x^4 - y^4))^2, \\ t_3 &= yx^{17} - 34y^5x^{13} + 34y^{13}x^5 - y^{17}x \end{aligned} \quad (5.16)$$

In the coordinates  $(t_1, t_2)$ , the metric  $\Omega_2(t)$  defined by the Hessian of  $\frac{12}{7}t_1$  is linear in  $t_2$  and leads to a regular QFPM with charge  $\frac{1}{3}$ . The resulting Frobenius manifold is of type  $I_2(3)$ .

5. The binary icosahedral  $\mathcal{I}$ : We fix the generators of the invariant ring

$$\begin{aligned} t_1 &= \frac{11}{30}(x^{11}y + 11x^6y^6 - xy^{11}), \\ t_2 &= x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}, \\ t_3 &= x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20} \end{aligned} \quad (5.17)$$

Fix  $(t_1, t_2)$  as coordinates, the Hessian of  $\frac{30}{11}t_1$  leads to a metric  $\Omega_2(t)$  linear in  $t_2$ . The regular QFPM formed by  $\Omega_2$  and  $\Omega_1 = \partial_{t_2}\Omega_2(t)$  leads to a Frobenius manifold of type  $I_2(5)$ .

## 6 Finite subgroups of $SL_3(\mathbb{C})$

Finite subgroups of  $SL_3(\mathbb{C})$  are classified into the families  $(\mathcal{A})$ ,  $(\mathcal{B})$ ,  $\dots$ ,  $(\mathcal{L})$  [24]. We treat them as representations of the corresponding groups and they are not reflection representations. Watanabe and Rotillon listed in [23] those subgroups where the invariant rings are complete intersections missing type  $(\mathcal{J})$  and  $(\mathcal{K})$ . These missing groups were recognized by Yau and Yu [24]. In the end, there is a total of 29 types of finite subgroups of  $SL_3(\mathbb{C})$  whose invariant rings are complete intersection and their sets of generators are known explicitly. We treat them as linear representations of finite groups and we apply Dubrovin's method. The set of generators is taken from [23] and we use the same numbering (1), (2),  $\dots$ , (27) of the 27 families of subgroups listed there.

Recall that to apply Dubrovin's method, we must find

$$\begin{aligned} &\text{a minimal degree invariant polynomial where the Hessian defines} \\ &\text{a flat contravariant metric.} \end{aligned} \quad (6.1)$$

This condition excluded the following subgroups

1. (17) which are of type  $(\mathcal{A})$ .
2. (3) – (8), (19) – (23) which are of type  $(\mathcal{B})$ .
3. (10), (24) and (25) which are of type  $(\mathcal{C})$ .
4. (13), (15), (16) and (27) which are of types  $(\mathcal{G})$ ,  $(\mathcal{L})$ ,  $(\mathbf{I})$  and  $(\mathcal{E})$ , respectively.
5. The groups  $(\mathcal{J})$  and  $(\mathcal{K})$  which are not considered in [23].

For the remaining family of subgroups, when condition (6.1) is satisfied, we use Lemma 2.5 to construct flat pencil of metric under appropriate choice of a set of invariant polynomials. We find natural Frobenius manifold structures of types  $A_3$ ,  $B_3$ ,  $H_3$ ,  $B_3^1$ , or the trivial  $T_3$ . In each case, we will mention the type of the resulting Frobenius structure and the corresponding flat coordinates. From Theorem 3.3, we know that ones the orbits space acquire one of these structures then it also possess the conjugate structure (see Example 4.6 and Example 3.4). Thus we will not mention explicitly the appearance of the natural conjugate Frobenius manifold structures.

- (1) This is a family of groups of type  $(\mathcal{A})$  depending on an integer  $m > 1$ . Complete set of generators of the invariant rings consists of  $x^m, y^m, z^m, xyz$ . The Hessian of  $xyz$  does not define a flat metric. Hence, condition (6.1) exclude the case  $m > 3$ .

For  $m = 2$ , we fix the invariant polynomials

$$u_1 = x^2 + y^2 + z^2, \quad u_2 = x^2y^2 + z^2y^2 + x^2z^2, \quad u_3 = (xyz)^2. \quad (6.2)$$

Then  $\{u_1, u_2, u_3\}$  can be identified with the set of generators of the invariant ring of the standard reflection representation of Coxeter groups of type  $B_3$ . Thus, applying Dubrovin's method, we get natural Frobenius manifold structures of types  $A_3$ ,  $B_3$  and  $B_3^1$ . We also get the natural trivial Frobenius manifold structure of type  $T_3$  using the setting of the family (2) given below. Thus, considering the conjugate structures and Frobenius manifolds obtained in Example 3.4, we proved that the orbits space has 8 different natural Frobenius manifold structures.

For  $m = 3$ , we fix the invariant polynomials

$$u_1 = x^3 + y^3 + z^3, \quad u_2 = x^3y^3 + y^3z^3 + z^3x^3, \quad u_3 = (xyz)^3. \quad (6.3)$$

The Hessian of  $u_1$  defines a contravariant flat metric  $\Omega_2$ . This metric and its Christoffel symbols are almost linear in each variable  $u_i$ . We can and will apply Lemma 2.5 and we get three regular QFPM. From QFPM  $(\Omega_2, \text{Lie}_{\partial_{u_3}} \Omega_2)$ , we get Frobenius manifold structure of type  $B_3$ . It has flat coordinates

$$t_1 = \frac{2}{9}u_1, \quad t_2 = -\frac{u_1^2 - 4u_2}{6\sqrt{2}}, \quad t_3 = \frac{7u_1^3}{216} - \frac{1}{6}u_2u_1 + u_3. \quad (6.4)$$

The QFPM  $(\Omega_2, \text{Lie}_{\partial_{u_2}} \Omega_2)$  leads to type  $A_3$ . It has flat coordinates

$$t_1 = \frac{1}{3}u_1, \quad t_2 = u_2 - \frac{1}{8}u_1^2, \quad t_3 = \sqrt{u_3}. \quad (6.5)$$

Finally we get Frobenius manifold structure of type  $B_3^1$  from the QFPM  $(\Omega_2, \text{Lie}_{\partial_{u_1}} \Omega_2)$ . Here the flat coordinates are

$$t_1 = u_1, \quad t_2 = u_2u_3^{-\frac{1}{4}}, \quad t_3 = \frac{4}{3}u_3^{\frac{1}{4}}. \quad (6.6)$$

- (2) This is a family of groups of type  $(\mathcal{B})$  depending on an integer  $m \geq 1$ . The polynomials  $x^{2m} + y^{2m}$ ,  $(xy)^2$ ,  $xyz(x^{2m} - y^{2m})$  and  $z^2$  form complete sets of generators for the invariant rings. Because of condition (6.1), we need only to consider  $m = 1$ . In this case, we fix the invariant polynomials

$$u_1 = x^2 + y^2 + z^2, u_2 = z^2, u_3 = x^2 y^2$$

The metric  $\Omega_2(u)$  defined by the Hessian of  $u_1$  and its Christoffel symbols are linear in each variable  $u_i$ . However, Lemma 2.5 is applicable only for  $u_2$ . The QFPM  $(\Omega_2, \text{Lie}_{\partial_{u_2}} \Omega_2)$  has degree 0 with  $\tau = u_1$ . It leads to a natural trivial Frobenius manifold structure of type  $T_3$ . Here the flat coordinates are

$$t_1 = \frac{1}{2}u_1, \quad t_2 = u_2 - \frac{1}{2}u_1, \quad t_3 = (-2u_3)^{\frac{1}{2}}. \quad (6.7)$$

- (9) This is a family of groups of type  $(\mathcal{C})$  depending on an integer  $m > 1$ . Complete sets of generators of the invariant rings consist of  $xyz, x^m + y^m + z^m, x^m y^m + x^m z^m + y^m z^m$ , and  $(x^m - y^m)(z^m - x^m)(y^m - z^m)$ . Here we get the same natural Frobenius manifold structure obtained for the family (1).
- (11) This is family of groups of type  $(\mathcal{C})$  depending on an integer  $m > 1$ . Complete sets of generators of the invariant rings consists of

$$u_1 = x^m + y^m + z^m, \quad u_2 = x^2 y^2 z^2, \quad u_3 = x^m y^m + y^m z^m + z^m x^m \quad (6.8)$$

and

$$u_4 = xyz(x^m - y^m)(z^m - x^m)(y^m - z^m). \quad (6.9)$$

Since the Hessian  $u_2$  does not define a flat metric, we consider only  $2 \leq m \leq 6$ . For  $m = 2$  we can use the same argument given for the family (1).

For  $3 \leq m \leq 6$ , the contravariant metric  $\Omega_2^{ij}$  defined by the Hessian of  $u_1$  and its Christoffel symbols are almost linear in  $u_1$  and  $u_3$  and we can apply Lemma 2.5 to both variables.

The FPM  $(\Omega_2, \text{Lie}_{\partial_{u_3}} \Omega_2)$  is regular quasihomogeneous of degree  $\frac{1}{2}$  with  $\tau = u_1$ . We can fix the flat coordinates

$$t_1 = \frac{m-1}{2m}u_1, \quad t_2 = u_2^{\frac{m}{4}}, \quad t_3 = u_3 - \frac{1}{8}u_1^2. \quad (6.10)$$

The resulting natural Frobenius manifold structure is a polynomial of type  $A_3$ .

Similarly, the FPM  $(\Omega_2, \text{Lie}_{\partial_{u_1}} \Omega_2)$  is regular quasihomogeneous of degree 0 with  $\tau = \frac{m-1}{m}u_1$ . We can fix the flat coordinates

$$t_1 = u_1, \quad t_2 = u_2^{\frac{m}{8}}, \quad t_3 = u_3 u_2^{-\frac{m}{8}}. \quad (6.11)$$

We get a natural Frobenius manifold structure of type  $B_3^1$ .

- (12) This is a group of type  $(\mathcal{F})$ . A complete set of generators of the invariant ring consists of

$$u_1 = (x^3 + y^3 + z^3)^2 - 12(x^3 y^3 + y^3 z^3 + z^3 x^3), \quad u_2 = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3), \quad (6.12)$$

$$u_3 = (xyz)^4 + 216(xyz)^3(x^3 + y^3 + z^3), \quad u_4 = ((x^3 + y^3 + z^3)^2 - 18(xyz)^2)^2.$$

The FPM  $(\Omega_2^{ij}, \text{Lie}_{\partial_{u_3}} \Omega_2^{ij})$  is regular quasihomogeneous of degree  $d = \frac{1}{2}$  with  $\tau = \frac{5}{12}u_1$ . Flat coordinates are given by

$$t_1 = \frac{5}{12}u_1, \quad t_2 = 10\sqrt{\frac{62}{41}}u_2, \quad t_3 = u_3 - \frac{847}{1312}u_1^2. \quad (6.13)$$

The resulting natural Frobenius manifold structure is of type  $A_3$ .

(14) This is a group of type  $(\mathcal{H})$  and a minimal set of generators of the invariant ring consists of

$$u_1 = x^2 + yz, \quad u_2 = 8yzx^4 - 2y^2z^2x^2 - (y^5 + z^5)x + y^3z^3,$$

and

$$u_3 = y^{10} + 6z^5y^5 + 20x^2z^4y^4 - 160x^4z^3y^3 + 320x^6z^2y^2 + z^{10} - 4x(y^5 + z^5)(32x^4 - 20yzx^2 + 5y^2z^2).$$

The Hessian of  $10u_1$  leads to a regular QFPM  $(\Omega_2, \text{Lie}_{\partial_{u_3}} \Omega_2)$  of degree  $d = \frac{4}{5}$  with  $\tau = \frac{1}{10}u_1$ . By fixing the flat coordinates

$$t_1 = \frac{1}{10}u_1, \quad t_2 = \sqrt{2}u_2 - \sqrt{2}u_1^3, \quad t_3 = 14u_1^5 - 20u_2u_1^2 + u_3,$$

we arrive to a natural polynomial Frobenius structure of type  $H_3$ .

(18) This is a family of groups of type  $(\mathcal{B})$  depending on integers  $p \geq 1$  and  $q \geq 2$ . A complete sets of generators of the invariant rings consists of  $(x^{2pq} + y^{2pq}), (xy)^{2q}, (xyz)^2, (x^{2pq} - y^{2pq})xyz, z^{2q}$ . The Hessian of  $(xyz)^2$  does not define a flat metric. From condition (6.1), we consider only the two cases:  $p = 1$  but  $q = 2$  or  $q = 3$ . In both cases we get 3 types of natural Frobenius manifold structures. The first natural Frobenius structure is of type  $T_3$ . It has the flat coordinates

$$t_1 = \frac{2q-1}{2q}(x^{2q} + y^{2q} + z^{2q}), \quad t_2 = \frac{-1}{2}(x^{2q} + y^{2q}), \quad t_3 = \left(\frac{2-4q}{q}xy\right)^{\frac{q}{2}}. \quad (6.14)$$

The corresponding regular QFPM is  $(\Omega_2, \text{Lie}_{\partial_{t_2}} \Omega_2)$  with  $\tau = t_1$  where  $\Omega_2$  defined by the Hessian of  $t_1$ . Let us fix

$$u_1 = x^{2q} + y^{2q} + z^{2q}, \quad u_2 = (xyz)^2, \quad u_3 = -2x^{2q}y^{2q} - 2x^{2q}z^{2q} - 2y^{2q}z^{2q}. \quad (6.15)$$

Then the second natural Frobenius manifold structure is of type  $B_3^1$ . It has the flat coordinates

$$t_1 = \frac{2q-1}{2q}u_1, \quad t_2 = u_2^{\frac{q}{4}}, \quad t_3 = u_3u_2^{\frac{-q}{4}} \quad (6.16)$$

The corresponding regular QFPM is  $(\Omega_2^1, \text{Lie}_{\partial_{t_1}} \Omega_2)$  has degree 0 with  $\tau = t_1$ . Finally, we get natural Frobenius manifold structure of type  $A_3$  having the flat coordinates

$$t_1 = \frac{2q-1}{4q}u_1, \quad t_2 = \frac{2\sqrt{2q-1}}{\sqrt{q}}u_2^{\frac{q}{2}}, \quad t_3 = u_3 + \frac{1}{4}u_1^2. \quad (6.17)$$

The corresponding regular QFPM  $(\Omega_2, \text{Lie}_{\partial_{t_3}} \Omega_2)$  is of degree  $\frac{1}{2}$  with  $\tau = t_1$ .

(26) This is a family of groups of type  $(\mathcal{C})$  depending on even integer  $m \geq 2$ . The set of minimal generators of invariant ring has

$$x^{3m} + y^{3m} + z^{3m}, (xyz)^2, x^{2m}y^m + x^my^{2m} + y^{2m}z^m + y^mz^{2m} + z^{2m}x^m + z^mx^{2m}, \\ xyz(x^m - y^m)(y^m - z^m)(z^m - x^m), (x^m - y^m)^2(y^m - z^m)^2(z^m - x^m)^2.$$

The only possible case under condition (6.1) is when  $m = 2$ . In this case we get a natural Frobenius manifold of type  $A_3$ . It has the flat coordinates

$$t_1 = \frac{5}{12}(x^6 + y^6 + z^6), \quad t_2 = \sqrt{\frac{20}{3}}(xyz)^3, \quad t_3 = x^{12} + y^{12} + z^{12} - \frac{3}{4}(x^6 + y^6 + z^6)^2.$$

Here,  $\Omega_2$  is defined by the Hessian of  $\frac{12}{5}t_1$  and the corresponding regular QFPM  $(\Omega_2, \text{Lie}_{\partial_{t_3}} \Omega_2)$  is of degree  $d = \frac{1}{2}$  with  $\tau = t_1$ .

On the other hand, the Hessian of  $\frac{5}{6}t_1$  leads to a regular QFPM  $(\Omega_2, \text{Lie}_{\partial_{t_1}} \Omega_2)$  of degree 0 with  $\tau = t_1$ . In this case the flat coordinates are

$$\begin{aligned} t_1 &= \frac{5}{6}(x^6 + y^6 + z^6), \quad t_2 = (xyz)^{\frac{3}{2}}, \\ t_3 &= \frac{-5}{6}(xyz)^{\frac{-1}{2}} \left( x^{12} + y^{12} + z^{12} - \frac{3}{4}(x^6 + y^6 + z^6)^2 \right). \end{aligned} \tag{6.18}$$

The resulting natural Frobenius manifold structure is of type  $B_3^1$ .

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## Data Availability

Non applicable.

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