Dicyclic groups and Frobenius manifolds

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Abstract

The orbits space of an irreducible representation of a finite group is a variety whose coordinate ring is finitely generated by homogeneous invariant polynomials. Boris Dubrovin showed that the orbits spaces of the reflection groups acquire the structure of polynomial Frobenius manifolds. Dubrovin's method to construct examples of Frobenius manifolds on orbits spaces was carried for other linear representations of discrete groups which have in common that the coordinate rings of the the orbits spaces are polynomial rings. In this article, we show that the orbits space of an irreducible representation of a Dicyclic group acquire two structures of Frobenius manifolds. The coordinate ring of this orbits space is not a polynomial ring.

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Contents

1 Introduction

The notion of Frobenius manifold was introduced by Boris Dubrovin as a geometric realization of a potential $\mathbb{F}$ satisfying a system of partial differential equations known in topological fields theory as WDVV equations [6]. Beside topological fields theory, Frobenius manifolds appear in many fields like invariant theory, integrable systems, quantum cohomology and singularity theory. This article contributes to the relation between Frobenius manifolds and invariant theory.

Let W be a finite group of linear transformations acting on a complex vector space V of dimention r. Then the orbits space $M = V/\mathrm{W}$ of this group is a variety whose coordinate ring is the ring of invariant polynomials $\mathbb{C}[V]^{\mathrm{W}}$. The ring $\mathbb{C}[V]^{\mathrm{W}}$ is finitely generated by homogeneous polynomials. If $f_1, f_2, \ldots, f_m$ is a set of such generators then $m \geq r$ and the relation between them is called syzygies. The set of generators are not unique, nor are their degrees [11],[4].

An element $w \in \mathrm{W}$ is called a reflection if it fixes a subspace of V of codimention one pointwise. The group W is called a complex reflection group if it is generated by reflections. Then Shephard-Todd-Chevalley theorem states that W is a reflection group if and only if the invariant ring $\mathbb{C}[V]^{\mathrm{W}}$ is a polynomial ring [11], i.e. it is generated by r algebraically independent homogeneous polynomials (so there are no syzygies). Furthermore, when W is a reflection group, the degrees of such a set generators of $\mathbb{C}[V]^{\mathrm{W}}$ are uniquely specified by the group and we refer to them as the degrees of W.

Let us assume W is a Shephard group, i.e. a symmetry group of a regular complex polytope. Then W is a reflection group. Let $f_1, f_2, \dots, f_r$ be a set of algebraically independent homogeneous generators of $\mathbb{C}[V]^{\mathrm{W}}$. We assume that degree f_i is less than or equal degree f_j when $i < j$. Then the inverse of the Hessian of f_1 defines a flat metric $(\cdot,\cdot)_2$ on T^*M [12]. There is another flat metric $(\cdot,\cdot)_1$ on T^*M, which was studied initially by K. Saito ([13], [14]), defined as the Lie derivative of $(\cdot,\cdot)_2$ along the vector field $e = \partial_{f_r}$. The two metrics form what is called a flat pencil of metrics (more details are given below). Dubrovin used the properties of this flat pencil of metrics to construct polynomial Frobenius manifolds [10] (see [7] and [15] for the case of Coxeter groups). This article is about applying Dubrovin's method for other finite linear groups than Shephard groups.

Dubrovin's method to construct Frobenius manifolds, through finding flat pencils of metrics on orbits spaces, was carried out for infinite linear groups like extended affine Weyl groups [8], [9], Jacobi groups [3] and recently a new extension of affine Weyl groups [16]. They all have in common that the invariant rings are polynomial rings. Moreover, even when considering a generalization of Frobenius manifold structure on orbits spaces, many results was obtained under the assumption that the invariant ring is a polynomial ring [2]. Then it is natural question to ask about applying Dubrovin's method on orbits spaces of finite non-reflection groups.

In this article we apply Dubrovin's method and construct Frobenius manifolds on orbits spaces of Dicyclic groups. The resulting Frobenius manifolds can be obtained by using an ad-hoc procedure, but it is fascinating to find them on orbits spaces of some group. Precisely, we will show that the orbits space of the Dicyclic group of order $4n$ is endowed with two structure of Frobenius manifolds which up to scaling has the following potential

$$\mathbb{F}(z_1, z_2) = z_1^k + \frac{1}{2} z_2^2 z_1, \ k = \frac{3-d}{1-d} \tag{1.1}$$

where d is $\frac{2+\sqrt{3}n}{\sqrt{3}n}$ or $\frac{2-\sqrt{3}n}{\sqrt{3}n}$.

To make the article as self-contained as possible, we review in next section the definition of Frobenius manifold and its relation with the theory of flat pencils of metrics. In the last section we obtain the promised Frobenius manifolds by direct calculations.

2 Preliminaries

2.1 Frobenius manifolds

A Frobenius algebra is a commutative associative algebra with unity e and an invariant nondegenerate bilinear form $< \cdot, \cdot >$. A **Frobenius manifold** is a manifold M with a smooth structure of a Frobenius algebra on the tangent space $T_t M$ at any point $t \in M$ with certain compatibility conditions [6]. Globally, we require the metric $< \cdot, \cdot >$ to be flat and the unity vector field e to be covariantly constant with respect to it. In the flat coordinates $(t^1, ..., t^r)$ where $e = \frac{\partial}{\partial t^r}$ the compatibility conditions imply that there exists a function $\mathbb{F}(t^1, ..., t^r)$ such that

$$\eta_{ij} =< \partial_{t^i}, \partial_{t^j} >= \partial_{t^r} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

and the structure constants of the Frobenius algebra are given by

$$C_{ij}^k = \sum_p \Omega_1^{kp} \partial_{t^p} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

where Ω_1^{ij} denote the inverse of the matrix η_{ij}. In this work, we consider Frobenius manifolds where the quasihomogeneity condition takes the form

$$\sum_{i=1}^{r} d_i t^i \partial_{t^i} \mathbb{F}(t) = (3-d)\,\mathbb{F}(t); \qquad d_r = 1. \tag{2.1}$$

This condition defines **the degrees** d_i and **the charge** d of the Frobenius structure. The associativity of the Frobenius algebra implies that the potential $\mathbb{F}(t)$ satisfies a system of partial differential equations which appears in topological field theory and is called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations:

$$\sum_{k,p} \partial_{t^i}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\ \Omega_1^{kp}\ \partial_{t^p}\partial_{t^q}\partial_{t^n}\mathbb{F}(t) = \sum_{k,p} \partial_{t^n}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\ \Omega_1^{kp}\ \partial_{t^p}\partial_{t^q}\partial_{t^i}\mathbb{F}(t), \quad \forall i,j,q,n. \tag{2.2}$$

Detailed information about Frobenius manifolds and related topics can be found in [6].

2.2 Flat pencil of metrics and Frobenius manifolds

In this section we review the relation between the geometry of flat pencil of metrics and Frobenius manifolds. See [5] for details.

Let M be a smooth manifold of dimension r. A symmetric bilinear form $(\cdot,\cdot)$ on T^*M is called a **contravariant metric** if it is invertible on an open dense subset $M_0 \subset M$. In local coordinates $(u^1,...,u^r)$, if we set

$$\Omega^{ij}(u) = (du^i, du^j);\ i,j = 1,...,r. \tag{2.3}$$

Then the inverse matrix $\Omega_{ij}(u)$ of $\Omega^{ij}(u)$ determines a metric $<\cdot,\cdot>$ on TM_0. We define the **contravariant Christoffel symbols** Γ_k^{ij} of $(\cdot,\cdot)$ by $\Gamma_k^{ij} := -\sum_s \Omega^{is}\Gamma_{sk}^j$ where Γ_{sk}^j are the Christoffel symbols of $<\cdot,\cdot>$. We say the metric $(\cdot,\cdot)$ is flat if $<\cdot,\cdot>$ is flat.

Let $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be two contravariant flat metrics on M and denote their Christoffel symbols by $\Gamma_{1;k}^{ij}(u)$ and $\Gamma_{2;k}^{ij}(u)$ respectively. We say $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ form a **flat pencil of metrics** if $(\cdot,\cdot)_\lambda := (\cdot,\cdot)_1 + \lambda(\cdot,\cdot)_2$ defines a flat metric on T^*M for a generic λ and its Christoffel symbols are given by $\Gamma_{\lambda;k}^{ij}(u) = \Gamma_{2;k}^{ij}(u) + \lambda\Gamma_{1;k}^{ij}(u)$.

Let $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be two contravariant metrics on M and denote their matrices by $\Omega_1^{ij}(u)$ and $\Omega_2^{ij}(u)$, respectively, in some coordinates $(u^1,\ldots,u^r)$. Suppose that they form a flat pencil of metrics. This flat pencil of metrics is called **quasihomogeneous of degree** d if there exists a function τ on M such that the vector fields

$$\begin{aligned} E &:= \nabla_2\tau, \quad E^i = \sum_s \Omega_2^{is}\partial_s\tau \\ e &:= \nabla_1\tau, \quad e^i = \sum_s \Omega_1^{is}\partial_s\tau \end{aligned} \tag{2.4}$$

satisfy the following relations

$$[e,E] = e, \quad \mathrm{Lie}_E(\ ,\ )_2 = (d-1)(\ ,\ )_2, \quad \mathrm{Lie}_e(\ ,\ )_2 = (\ ,\ )_1, \quad \mathrm{Lie}_e(\ ,\ )_1 = 0.$$

Here Lie_X denote the Lie derivative along a given vector field X. In addition, the quasihomogeneous flat pencil of metrics is called **regular** if the (1,1)-tensor $R_i^j = \frac{d-1}{2}\delta_i^j + \nabla_{1i}E^j$ is nondegenerate on M.

The following theorem due to Dubrovin gives a connection between the geometry of Frobenius manifolds and flat pencils of metrics.

Theorem 2.1. *[5] A quasihomogeneous regular flat pencil of metrics of degree d on a manifold M defines a Frobenius structure on M of charge d.*

Let us assume the flat pencil of metrics on M is regular quasihomogeneous of degree d. Let $(t^1, \dots, t^r)$ be flat coordinates of $(\cdot,\cdot)_1$ where $\tau = t^1$, $e = \partial_{t^r}$ and $E = \sum_i d_i t^i \partial_{t^i}$. Let η_{ij} denote the inverse of $\Omega_1^{ij}(t)$. Then it turns out that the potential $\mathbb{F}(t^1, \dots, t^r)$ is obtained from the equations

$$\begin{aligned} \frac{\partial^2 \mathbb{F}}{\partial t^i \partial t^j} &= \sum_{k,l} \frac{1}{d-1+d_k+d_l} \eta_{ik}\eta_{jl}\Omega_2^{kl}(t) \\ \frac{\partial \mathbb{F}}{\partial t^i} &= \frac{1}{(3-d-d_i)} \sum d_k t^k \frac{\partial^2 \mathbb{F}}{\partial t^k \partial t^i} \\ \mathbb{F}(t) &= \frac{1}{3-d} \sum d_k t^k \frac{\partial \mathbb{F}}{\partial t^k} \end{aligned} \tag{2.5}$$

It is well known that from a Frobenius manifold we always have a flat pencil of metrics but it does not necessarily satisfy the regularity condition [5].

3 Dicyclic groups

Let n be a natural number greater that 1 and W be the matrix group generated by

$$\sigma := \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \ \alpha := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.1}$$

where ξ is a primitive $2n$-th root of unity. Then σ and α satisfy the relations

$$\sigma^{2n} = 1, \alpha^2 = \sigma^n, \alpha^{-1}\sigma\alpha = \sigma^{-1}. \tag{3.2}$$

Thus W is isomorphic to the dicyclic group of order $4n$. The invariant ring of W is generated by the following homogeneous polynomials [11]

$$u_1 = x_1^2 x_2^2, \ u_2 = x_1^{2n} + x_2^{2n}, \ u_3 = x_1 x_2 (x_1^{2n} - x_2^{2n}) \tag{3.3}$$

subject to the relation

$$u_3^2 - u_1 u_2^2 + 4u_1^{n+1} = 0. \tag{3.4}$$

The orbits space M of W is a variety isomorphic to the hypersurface T defined as the zero set of equation (3.4) in $\mathbb{C}^3$. Consider equation (3.4) as a quadratic equation in u_3. Then any point p out of the discriminant locus has small neighbourhood U_p where u_1 and u_2 act as coordinates. In what follows we assume that we fix such open set $U \subset V$ with coordinates (u_1, u_2) and we omit the subscript p.

Let h be the Hessain matrix of u_1, i.e. $h_{ij} = \frac{\partial^2 u_1}{\partial x_i \partial x_j}$ and let h^{-1} denotes its inverse. Then, by direct calculations, h^{-1} defines a flat contravariant metric $(\cdot,\cdot)_2$ on U. This metric, in the coordinates u_1 and u_2, is given as follows

$$(\cdot,\cdot)_2 = \begin{pmatrix} \frac{4}{3}u_1 & \frac{2n}{3}u_2 \\ \frac{2n}{3}u_2 & -\frac{2n^2}{3u_1}(u_2^2 - 6u_1^n) \end{pmatrix}; \quad (dt_i, dt_j)_2 = \sum_{k,l=1}^{2} \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} h_{kl}^{-1}. \tag{3.5}$$

Let e be a vector field of the form $f(u_1)\partial_{u_2}$, where $f(u_1)$ is any smooth function. Then, by direct calculations, the Lie derivative $(\cdot,\cdot)_1$ of $(\cdot,\cdot)_2$ along e forms with $(\cdot,\cdot)_2$ a flat pencil of metrics. This metric takes the value

$$(\cdot,\cdot)_1 = \begin{pmatrix} 0 & \frac{2}{3}(nf - 2u_1 f') \\ \frac{2}{3}(nf - 2u_1 f') & -\frac{4}{3u_1}(n^2 u_2 f + n u_1 u_2 f') \end{pmatrix}. \tag{3.6}$$

The guess for the vector field to take this from was inspired by [1]. In order to get a quasihomogeneous flat pencil of metrics, we need the Lie derivative of $(\cdot,\cdot)_1$ with respect to e to equal zero. This condition leads to the following differential equation for $f(u)$

$$2nu_1 f f' - 2u_1^2 (f')^2 + n^2 f^2 = 0 \tag{3.7}$$

which has two independent solutions

$$f_+ = u_1^{\frac{n}{2}\left(1+\sqrt{3}\right)} \text{ and } f_- = u_1^{\frac{n}{2}\left(1-\sqrt{3}\right)}. \tag{3.8}$$

Let us assume $e = f_+ \partial_{u_2} = u_1^{\frac{n}{2}\left(1+\sqrt{3}\right)} \partial_{u_2}$. Then

$$(dt_i, dt_j)_1 = \begin{pmatrix} 0 & -\frac{2n}{\sqrt{3}} u_1^{\frac{1}{2}\left(1+\sqrt{3}\right)n} \\ -\frac{2n}{\sqrt{3}} u_1^{\frac{1}{2}\left(1+\sqrt{3}\right)n} & -\frac{2}{3}\left(3+\sqrt{3}\right) n^2 u_1^{\frac{1}{2}\left(1+\sqrt{3}\right)n-1} u_2 \end{pmatrix}. \tag{3.9}$$

It turns out that the two metrics $(\cdot,\cdot)_2$ and $(\cdot,\cdot)_1$ form a quasihomogeneous flat pencil of metrics with degree

$$d = \frac{2+\sqrt{3}n}{\sqrt{3}n}. \tag{3.10}$$

In the notations of equations (2.4), we have $\tau = -\frac{\sqrt{3}}{2n} u_1$ and

$$E = -\frac{2}{\sqrt{3}n} u_1 \partial_{u_1} - \frac{1}{\sqrt{3}} u_2 \partial_{u_2}. \tag{3.11}$$

This flat pencil of metrics is also regular since the $(1,1)$-tensor R_i^j equals the nondegenerate matrix

$$\begin{pmatrix} -\frac{1}{\sqrt{3}n} & 0 \\ 0 & \frac{1-n}{\sqrt{3}n} \end{pmatrix}. \tag{3.12}$$

Flat coordinates for $(\cdot,\cdot)_1$ are obtained by setting

$$t_1 = -\frac{\sqrt{3}}{2n} u_1, \quad t_2 = u_2 u_1^{-\frac{1}{2}\left(1+\sqrt{3}\right)n}. \tag{3.13}$$

In these coordinates we get

$$(\cdot,\cdot)_2 = \begin{pmatrix} -\frac{2}{\sqrt{3}n} t_1 & t_2 \\ t_2 & 2^{1-\sqrt{3}n} 3^{\frac{1}{2}\left(\sqrt{3}n+1\right)} n^2 \left(-nt_1\right)^{-\sqrt{3}n-1} \end{pmatrix}, \quad (\cdot,\cdot)_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.14}$$

The potential $\mathbb{F}_+$ of the corresponding Frobenius manifold is

$$\mathbb{F}_+ = \frac{2^{-\sqrt{3}n} 3^{\frac{1}{2}\left(\sqrt{3}n+1\right)} \left(-nt_1\right)^{1-\sqrt{3}n}}{3n^2-1} + \frac{1}{2} t_1 t_2^2 \tag{3.15}$$

Let us take $e = f_- \partial_{u^2} = u_1^{\frac{n}{2}\left(1-\sqrt{3}\right)} \partial_{u_2}$. Then similar to above, we get a regular quasihomogenous flat pencil of metrics of degree

$$d = \frac{2-\sqrt{3}n}{\sqrt{3}n} \tag{3.16}$$

with $\tau = \frac{\sqrt{3}}{2n}u_1$. The resulting potential will be

$$\mathbb{F}_- = \frac{2^{\sqrt{3}n} 3^{\frac{1}{2}-\frac{\sqrt{3}n}{2}} (nt_1)^{\sqrt{3}n+1}}{3n^2-1} + \frac{1}{2}t_1t_2^2. \tag{3.17}$$

We repeat the calculation by taking (u_1, u_3) as coordinates instead of (u_1, u_2). It turns out that even though the middle steps may differ in values, the resulting Frobenius manifolds are exactly the same as those given by the potentials (3.15),(3.17).

We observe that Dubrovin computed by ad hoc procedure all possible potentials of 2-dimensional Frobenius manifolds [6]. The potentials found in this article, after scaling, are listed by Dubrovin in the form

$$\mathbb{F}(z_1, z_2) = z_1^k + \frac{1}{2}z_2^2 z_1,\ k = \frac{3-d}{1-d} \tag{3.18}$$

where d is $\frac{2+\sqrt{3}n}{\sqrt{3}n}$ or $\frac{2-\sqrt{3}n}{\sqrt{3}n}$. However, finding it by using the method of a flat pencil of metrics on an orbits space of a finite group that is not a reflection group is a surprising result.

The result reported in this article is a part of work in progress to apply Dubrovin's method on orbits spaces of finite groups to find new interesting examples of Frobenius manifolds. In future publications, we will consider irreducible representations of Coxeter groups which are not reflection representations [1].

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